Matrix-Valued Statistics for Detecting Complex Model Errors in State Estimation

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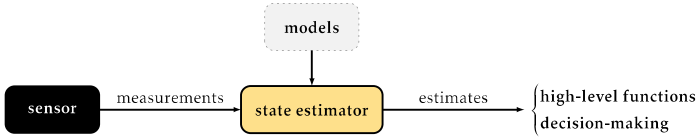
State Estimation



The figure illustrates a *state estimator*, e.g., an *extended Kalman filter* (EKF) which uses *sensor measurements* and a *state-space model* (SSM) to compute estimates of a state of interest.

State-Space Model

Let $\mathbf{x}_k$ be a state of interest at time k. A common SSM is

$$\mathbf{x}_{k+1} = \mathbf{F}_k\mathbf{x}_k + \mathbf{w}_k, \qquad \mathbf{w}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_k), \tag{1a}$$
$$\mathbf{y}_k = \mathbf{h}(\mathbf{x}_k) + \mathbf{v}_k, \qquad \mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_k), \tag{1b}$$

where $\mathbf{F}_k$ is a state transition matrix, $\mathbf{w}_k$ is process noise, $\mathbf{y}_k$ is a measurement, $\mathbf{h}(\cdot)$ is a measurement model, $\mathbf{v}_k$ is measurement noise, and $\mathcal{N}(\mathbf{0}, \Sigma)$ is a zero-mean Gaussian distribution with covariance Σ.

The Problem

The estimates are used in *high-level functions*, e.g., decision-making. To this end, it is crucial to emphasize that:

- ✗ advanced decision-making algorithms **require accurate models**,
- ✗ state-space models are **never 100% correct**, and
- ✗ **existing measures** are scalar-valued and are hence **unsufficient to address multivariate relationships**.

More sensitive and precise measures are needed!

The Normalized Innovation Squared

Key quantities in an EKF are the *innovation* $\tilde{\mathbf{y}}_k$ and *innovation covariance* $\mathbf{S}_k$ which, for the SSM in (1), are defined as

$$\tilde{\mathbf{y}}_k = \mathbf{y}_k - \mathbf{h}(\hat{\mathbf{x}}_{k|k-1}), \qquad \mathbf{S}_k = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k^{\mathsf{T}} + \mathbf{R}_k,$$

where $\hat{\mathbf{x}}_{k|k-1}$ is the predicted estimate, $\mathbf{P}_{k|k-1}$ the associated covariance, and $\mathbf{H}_k = \frac{\partial \mathbf{h}(\mathbf{x})}{\partial \mathbf{x}}|_{\mathbf{x}=\hat{\mathbf{x}}_{k|k-1}}$. If the SSM is correct, then

$$\tilde{\mathbf{y}}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{S}_k). \tag{2}$$

To test if (2) holds the following statistics is proposed:

The NIS Matrix

The *normalized innovation squared* (NIS) matrix is given by

$$\hat{\mathbf{\Pi}}_k = \frac{1}{K}\sum_{l=k-K+1}^{k} \mathbf{B}_l^{-1}\tilde{\mathbf{y}}_l\tilde{\mathbf{y}}_l^{\mathsf{T}}\mathbf{B}_l^{-\mathsf{T}}, \tag{3}$$

where K is the number of time steps used and $\mathbf{B}_l\mathbf{B}_l^{\mathsf{T}} = \mathbf{S}_l$.

Common practice today, for assessing (2), is to use the scalar NIS

$$\epsilon_k = \frac{1}{K}\sum_{l=k-K+1}^{k} \tilde{\mathbf{y}}_l^{\mathsf{T}}\mathbf{S}_l^{-1}\tilde{\mathbf{y}}_l = \mathrm{tr}(\hat{\mathbf{\Pi}}_k). \tag{4}$$

Statistics

Let m denote the dimensionality of $\tilde{\mathbf{y}}_k$. Given that (2) holds, both ϵ_k and $\hat{\mathbf{\Pi}}_k$ have well-defined statistics:

$$\hat{\mathbf{\Pi}}_k \sim \mathcal{W}_m(K, \mathbf{I}) \qquad \epsilon_k \sim \chi^2_{mK}$$

Wishart distribution
K degrees-of-freedom
covariance parameter $\mathbf{I}$

chi-square distribution
mK degrees-of-freedom

Of particular interest are the distributions for the smallest and largest eigenvalue, $\lambda_{\min}(\hat{\mathbf{\Pi}}_k)$ and $\lambda_{\max}(\hat{\mathbf{\Pi}}_k)$, of $\hat{\mathbf{\Pi}}_k$.

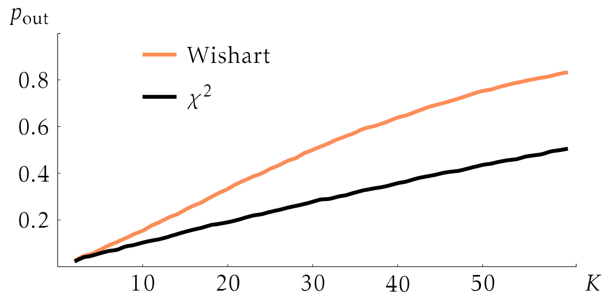
Resources

A publicly available github repository can be retrieved by the QR-code. The repository is connected to the paper "Matrix-Valued Measures and Wishart Statistics for Target Tracking Applications" and contains:

- exact distributions for $\lambda_{\min}(\hat{\mathbf{\Pi}}_k)$ and $\lambda_{\max}(\hat{\mathbf{\Pi}}_k)$,
- computional efficient approximative distributions for $\lambda_{\min}(\hat{\mathbf{\Pi}}_k)$ and $\lambda_{\max}(\hat{\mathbf{\Pi}}_k)$, and
- examples and applications.

Example

Assume that the SSM in (1) holds, but that an EKF uses the process noise covariance $\mathbf{Q}_k$, while the true process noise covariance is given by $\mathbf{Q}_k^0 \neq \mathbf{Q}_k$. The task is to detect this model mismatch using: (i) the NIS and χ^2 statistics, and (ii) the NIS matrix and Wishart eigenvalue statistics. The probability of detecting the model error, p_{out}, as a function of K is plotted below:



The NIS matrix is more sensitive to detect model errors!

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